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On the moment problem for non-positive distributions

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Abstract. Motivated by considerations arising from many-body quantum physics we consider the moment problem in the general case where the moments are finite real numbers. We present a well defined analytic procedure for the construction of an infinite set of exact solutions to the above problem and discuss several special cases.

1. Introduction

The problem of moments has a long history and has occupied the attention of many of the most eminent mathematicians. The theory they constructed is now encapsulated primarily in a very elegant book (Akhiezer 1965). The problems that have been studied most intensively are those concerned with the moments of some non-decreasing function $\sigma(x)$, which are defined by

$$s_k = \int_{-\infty}^{\infty} x^k \, d\sigma(x) \qquad k = 0, 1, 2, \ldots \tag{1.1}$$

where the integral is of the Lebesque–Stieltjes variety. It is required to find $\sigma(x)$ given the moments s_k. This problem will henceforth be called the positive-weight moment problem (PWMP) (Haydock 1976, Whitehead *et al* 1977, Whitehead and Watt 1981).

Very little attention, however, has been given to the possibility of finding a function, which is not necessarily non-decreasing and satisfies equation (1.1), provided the moments are known. We shall refer to this as the moment problem for non-positive distributions or as the negative-weight moment problem (NWMP). There is an intuitive implausibility about the NWMP and with the exception of a few sideglances in connection with Gaussian quadrature with negative-weight functions (Krylov 1962) there seems to exist no literature on it.

In a recent work (Flessas *et al* 1982) we have described a method for finding solutions to equation (1.1) for any set of s_k. The aim of the present paper is to extend the procedure introduced in our previous work and thereby construct an infinite set of solutions for equation (1.1) for arbitrary real moments s_k. In § 2 we provide the physical motivation for considering this kind of problem. In § 3 we carry out the solution of equation (1.1) for any given real s_k and in § 4 we make some comments concerning the solutions thus obtained and discuss several special cases. Finally in § 5 we summarise our results.

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2. Relation of the problem with quantum physics

Our interest in the problem of moments stems from its connection with the theory of renormalised or effective interactions in quantum many-body physics. Consider a quantum mechanical system whose time-independent Schrödinger equation in some basis is

$$H\psi = E\psi. \tag{2.1}$$

In general H will be a real symmetric matrix of very large (possibly infinite) order and ψ will be a correspondingly large vector; E is the energy of the system. These matrices and vectors may be formally partitioned thus

$$\begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{m}} \\ \psi_{\mathrm{e}} \end{pmatrix} = E\begin{pmatrix} \psi_{\mathrm{m}} \\ \psi_{\mathrm{e}} \end{pmatrix}. \tag{2.2}$$

In performing the partitioning we have it in mind that of the entire vector space inhabited by ψ there may be a relatively small portion which governs the main physics of the problem, the rest of the space contributing 'higher-order corrections'. In the nuclear shell model, for example, the partitioning is suggested by the apparent shell structure (which we must point out is not completely understood). The practice is usually to choose the largest *model space*, whence the notation ψ_{m} in equation (2.2), that can be handled conveniently and totally ignore the rest, which may therefore be called the *excluded space*, whence ψ_{e}.

In order to get agreement between theory and experiment it is then necessary to modify in some way the model space part, A, of the Hamiltonian matrix in equation (2.2). The resulting matrix is then called an *effective* Hamiltonian for the model space concerned. So far no really satisfying *a priori* method of obtaining the effective Hamiltonian has been found.

An alternative approach is to eliminate ψ_{e} from equation (2.2), thus expressing everything in terms of the model space part of ψ. The result is

$$\left(A + C\frac{1}{E-B}C^{\mathrm{T}}\right)\psi_{\mathrm{m}} = E\psi_{\mathrm{m}} \tag{2.3}$$

where the second term on the left may be thought of as a renormalisation of the 'bare' Hamiltonian A. Equation (2.3) is now a nonlinear eigenvalue problem. The discussion of practical methods for coping with equation (2.3) and the investigation of its relation with the linear eigenvalue problem, though important, are of no direct concern to us here. Of more relevance to the present investigation is the following observation. If the model space were chosen to be one dimensional, equation (2.3) becomes

$$A + c\frac{1}{E-B}c^{\mathrm{T}} = E \tag{2.4}$$

where c^{T} is the column vector into which the rectangular matrix C^{T} degenerates in these circumstances. Note that c^{T} is a vector entirely in the excluded space. The nonlinear term in equation (2.4) is thus the matrix element of the resolvent $(E-B)^{-1}$ in the vector c^{T}. The connection with the PWMP is now apparent. We may write (see also Whitehead *et al* 1977)

$$c\frac{1}{E-B}c^{\mathrm{T}}=|c|^2\cfrac{1}{E-\alpha_0-\cfrac{\beta_0^2}{E-\alpha_1-\cfrac{\beta_1^2}{E-\alpha_2-\dots}}} \tag{2.5}$$

where the α and β are the elements of the Jacobi matrix (Akhiezer 1965) associated with the moments

$$s_k=\frac{1}{|c|^2}cB^kc^{\mathrm{T}}=\sum_{i=1} m_i x_i^k \tag{2.6}$$

x_i being the eigenvalues of the real and symmetric matrix B, and

$$m_i=(e_ic^{\mathrm{T}}/|c|)^2>0. \tag{2.7}$$

Equation (2.7) is the scalar product of the ith eigenvector of B with the vector c^{T}. Due to equation (2.7) it is obvious that the weight function in equation (2.6) is positive. The theory of moments assures us of the convergence of the continued fraction in equation (2.5) everywhere in the complex E plane except at the eigenvalues of B (which are on the real axis). The important point is that the standard theory of the PWMP is applicable (Whitehead *et al* 1977). In practice, in fact, we are given the s_k and in the context of the above theory we can calculate both the x_i and the m_i. For the sake of completeness we note that equation (2.6) implies that $\sigma(x)$ in equation (1.1) is sectionally constant and possesses discontinuities m_i at x_i; this is just one of the (in general) infinite set of solutions to equation (1.1), where an *infinite* set of moments is given.

Now, let us expand the model space to two dimensions. The coupling matrix C^{T} in equation (2.3) consists then of two column vectors c_1^{T} and c_2^{T} and so equation (2.3) becomes

$$\begin{pmatrix} A_{11}+c_1(E-B)^{-1}c_1^{\mathrm{T}} & A_{12}+c_1(E-B)^{-1}c_2^{\mathrm{T}} \\ A_{12}+c_2(E-B)^{-1}c_1^{\mathrm{T}} & A_{22}+c_2(E-B)^{-1}c_2^{\mathrm{T}} \end{pmatrix}\begin{pmatrix}\psi_{\mathrm{m}1}\\ \psi_{\mathrm{m}2}\end{pmatrix}=E\begin{pmatrix}\psi_{\mathrm{m}1}\\ \psi_{\mathrm{m}2}\end{pmatrix}. \tag{2.8}$$

It is clear that the nonlinear part of the off-diagonal terms are matrix elements of the resolvent between two different vectors and, therefore, that nonlinear contributions in the matrix in equation (2.8) may lead us to a NWMP in general. But expanding the model space from one dimension to two cannot add to the inherent difficulty of the problem. The most plausible expansion would consist of choosing the 2×2 matrix A so that its eigenvalues most closely approximate the two lowest eigenenergies of the system, and, hence, in the vicinity of the ground state equation (2.8) ought to be much better than equation (2.4) where the entire perturbation comes in through the resolvent. We thus have an apparent paradox. There are two ways out: either the whole effective interaction idea is useless and we are indulging in idle speculation, or the NWMP, at least insofar as it impinges upon off-diagonal matrix elements of resolvents, has some unsuspected features which translate into sensible solutions for equation (2.8). The success of calculations with effective Hamiltonians in various many-body problems shows that the first alternative is probably wrong and so we are left with the second.

Having now described the physical basis for our interest in the NWMP we proceed to show that in principle solutions of the problem are possible.

3. Formal solution to the NWMP

In this section we shall show that given *any* sequence of finite real numbers s_k, $k=0, 1, 2, \ldots, 2n-1$, it is possible to find a piece-wise constant function $\sigma(x)$ which has, in general, $2n$ points of discontinuity and whose moments

$$s_k = \int_{-\infty}^{\infty} x^k \, \mathrm{d}\sigma(x) \tag{3.1}$$

are equal to the given numbers. This function is not unique and we construct it in such a way that its degree of arbitrariness is clear.

There is a well known theorem according to which any function of bounded variation can be expressed in infinitely many ways as the difference of two non-decreasing functions of bounded variation (Hobson 1957). We begin, therefore, with the conjecture that there exists a $\sigma(x)$ satisfying equation (3.1). Since

$$\sigma(\infty) - \sigma(-\infty) = s_0 < \infty \tag{3.2}$$

the theorem applies here. Consequently we may write for $\sigma(x)$

$$\sigma(x) = \sigma^+(x) - \sigma^-(x) \tag{3.3}$$

where $\sigma^+(x)$ and $\sigma^-(x)$ are both non-decreasing functions of x and of bounded variation. To solve our problem it is clearly not sufficient to choose an arbitrary non-decreasing σ^+ (say) because σ^- may not turn out to be non-decreasing; they have to be determined simultaneously. Thus from equations (3.1)–(3.3) we have

$$s_k = \int_{-\infty}^{\infty} x^k \, \mathrm{d}\sigma^+(x) - \int_{-\infty}^{\infty} x^k \, \mathrm{d}\sigma^-(x) = s_k^+ - s_k^- \tag{3.4}$$

where s_k^+ and s_k^- are the moments belonging to $\sigma^+(x)$ and $\sigma^-(x)$ respectively. Consider now the quadratic form

$$Q = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k = \sum_{i,k=0}^{n-1} s_{i+k}^+ x_i x_k - \sum_{i,k=0}^{n-1} s_{i+k}^- x_i x_k \tag{3.5}$$

where use of equation (3.4) has been made. Since $\sigma^+(x)$ and $\sigma^-(x)$ are both non-decreasing and are assumed (as in the standard PWMP) to possess n points of increase each, we know that (Akhiezer 1965)

$$\left.\begin{aligned} Q_1 &= \sum_{i,k=0}^{n-1} s_{i+k}^+ x_i x_k > 0 \\ Q_2 &= \sum_{i,k=0}^{n-1} s_{i+k}^- x_i x_k > 0 \end{aligned}\right\} \quad \text{for any real } x. \tag{3.6}$$

The existence of $\sigma(x)$ thus depends on whether it is possible to make the decomposition of s_k in equation (3.4) while maintaining the positivity conditions (3.6) which are necessary *and* sufficient. We shall show in the following that this is possible.

The argument is a little bit clearer if we use Dirac notation. Let us define an orthogonal basis $|v_i\rangle$, $i=0, \ldots, n-1$, and an operator $\mathscr{A}$ such that

$$\langle v_i | \mathscr{A} | v_k \rangle = s_{i+k}. \tag{3.7}$$

Then

$$Q=\langle x|\mathscr{A}|x\rangle=(x_0 x \ldots x_{n-1})\begin{pmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-2}\end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1}\end{pmatrix} \tag{3.8}$$

with

$$|x\rangle=\sum_{i=0}^{n-1} x_i|v_i\rangle. \tag{3.9}$$

We next make a principal axis transformation on Q so that

$$Q=\langle x|\mathscr{A}|x\rangle=\sum_{i=0}^{n-1}\lambda_i x_i'^2 \tag{3.10}$$

where $x_i'=\langle e_i|x\rangle$ and $|e_i\rangle$ is the eigenvector of $\mathscr{A}$ corresponding to the eigenvalue λ_i. Since the matrix (3.7) is real and symmetric all the λ_i are real. It is now possible to rearrange equation (3.10) so that positive and negative contributions to the sum are written separately. Alternatively, we may define new operators $\mathscr{A}^+$ and $\mathscr{A}^-$ such that

$$\mathscr{A}^+|e_i\rangle=\begin{cases}\lambda_i|e_i\rangle & \lambda_i\geqslant 0\\ 0 & \lambda_i<0\end{cases} \tag{3.11}$$

$$\mathscr{A}^-|e_i\rangle=\begin{cases}0 & \lambda_i\geqslant 0\\ |\lambda_i||e_i\rangle & \lambda_i<0.\end{cases} \tag{3.12}$$

Note that any zero eigenvalues are grouped with the positive ones. The reason for this will become clear in § 4. Then according to equations (3.10)–(3.12)

$$\mathscr{A}=\mathscr{A}^+-\mathscr{A}^- \qquad Q=\langle x|\mathscr{A}|x\rangle=\langle x|\mathscr{A}^+|x\rangle-\langle x|\mathscr{A}^-|x\rangle=Q^+-Q^- \tag{3.13}$$

with

$$Q^+=\langle x|\mathscr{A}^+|x\rangle=\sum_{\lambda_k\geqslant 0}^{n-1}\langle x|e_k\rangle^2\lambda_k \qquad Q^-=\langle x|\mathscr{A}^-|x\rangle=\sum_{\lambda_k<0}^{n-1}\langle x|e_k\rangle^2|\lambda_k| \tag{3.14}$$

where equations (3.10) and (3.13) have been taken into account. Equation (3.14) shows that $Q^\pm\geqslant 0$. Further

$$Q^\pm=\langle x|\mathscr{A}^\pm|x\rangle=\sum_{i,k=0}^{n-1}\langle x|v_i\rangle\langle v_i|\mathscr{A}^\pm|v_k\rangle\langle v_k|x\rangle=\sum_{i,k=0}^{n-1}s_{ik}^\pm x_i x_k \tag{3.15}$$

$$\begin{aligned} s_{ik}^+&=\langle v_i|\mathscr{A}^+|v_k\rangle=\sum_{\lambda_1\geqslant 0}\langle v_i|e_l\rangle\lambda_l\langle e_l|v_k\rangle=s_{ki}^+\\ s_{ik}^-&=\langle v_i|\mathscr{A}^-|v_k\rangle=\sum_{\lambda_1<0}\langle v_i|e_l\rangle|\lambda_l|\langle e_l|v_k\rangle=s_{ki}^-.\end{aligned} \tag{3.16}$$

Obviously $s_{ii}^\pm\geqslant 0$. On using equations (3.7), (3.13) and (3.16) we find

$$s_{i+k}=\langle v_i|\mathscr{A}|v_k\rangle=s_{ik}^+-s_{ik}^-. \tag{3.17}$$

Since $Q^\pm\geqslant 0$, and bearing in mind that our objective is the decomposition (3.5), it remains only to show that $s_{ik}^\pm$ can be related to the moments of some positive distributions. As matters stand they are just numbers.

We observe now that equation (3.17) retains its validity if we add an arbitrary number a_{ik} to both s^{+}_{ik} and s^{-}_{ik}. In the rest of this section we shall show that this can be done in such a way that the new numbers $s^{\pm}_{ik}+a_{ik}$ fulfil the conditions necessary and sufficient for them to be the moments of a positive distribution. These conditions are (Akhiezer 1965)

$$
\left.\begin{aligned}
Q_1 &= \sum_{i,k=0}^{n-1} (s^{+}_{ik}+a_{ik})x_i x_k > 0 \\
Q_2 &= \sum_{i,k=0}^{n-1} (s^{-}_{ik}+a_{ik})x_i x_k > 0
\end{aligned}\right\} \quad \text{for any real } x \tag{3.18}
$$

and that

$$s^{\pm}_{ik}+a_{ik}=s^{\pm}_{i+k}. \tag{3.19}$$

Clearly owing to equations (3.13) and (3.18) $Q=Q_1-Q_2$. We have, therefore, to determine the a_{ik} so that equations (3.18)–(3.19) are satisfied. From equations (3.16) and (3.19) it follows that we must require

$$a_{ik}=a_{ki}. \tag{3.20}$$

A necessary and sufficient condition for equation (3.18) to be valid is (Akhiezer 1965)

$$
D^{\pm}_m = \begin{vmatrix}
a_{00}+s^{\pm}_{00} & a_{01}+s^{\pm}_{01} & \cdots & a_{0m-1}+s^{\pm}_{0m-1} \\
a_{01}+s^{\pm}_{01} & a_{11}+s^{\pm}_{11} & \cdots & a_{1m-1}+s^{\pm}_{1m-1} \\
\vdots & \vdots & & \vdots \\
a_{0m-1}+s^{\pm}_{0m-1} & a_{1m-1}+s^{\pm}_{1m-1} & \cdots & a_{m-1m-1}+s^{\pm}_{m-1m-1}
\end{vmatrix} > 0
$$

$$m=1,\ldots,n. \tag{3.21}$$

Before examining the general case we shall illustrate the procedure in the case of $n=4$. We start with equation (3.19). Thus we obtain

$$D^{+}_1 = s^{+}_{00}+a_{00}>0 \tag{3.22}$$

$$s^{+}_{11}+a_{11}=s^{+}_{02}+a_{02} \tag{3.23}$$

$$s^{+}_{03}+a_{03}=s^{+}_{12}+a_{12} \tag{3.24}$$

$$s^{+}_{22}+a_{22}=s^{+}_{13}+a_{13} \tag{3.25}$$

and also

$$D^{-}_1 = s^{-}_{00}+a_{00}>0 \tag{3.22a}$$

$$s^{-}_{11}+a_{11}=s^{-}_{02}+a_{02} \tag{3.23a}$$

$$s^{-}_{03}+a_{03}=s^{-}_{12}+a_{12} \tag{3.24a}$$

$$s^{-}_{22}+a_{22}=s^{-}_{13}+a_{13}. \tag{3.25a}$$

Now, equations (3.22) and (3.22a) hold if we choose

$$a_{00}>\max(-s^{+}_{00}, s^{-}_{00}). \tag{3.26}$$

Furthermore, take any real a_{01}; then a_{11} is determined from equation (3.21)

$$D^{\pm}_2>0. \tag{3.27}$$

Having thus obtained a_{11} we may find a unique value for a_{02} from equation (3.23). Next choose an arbitrary real a_{12}. Then a_{03} follows uniquely from equation (3.24). To compute a_{22} we consider

$$D_3^{\pm} > 0. \tag{3.28}$$

The parameter a_{13} is fixed therefore by equation (3.25). Finally, a_{23} is completely arbitrary and a_{33} is chosen subject to the condition

$$D_4^{\pm} > 0. \tag{3.29}$$

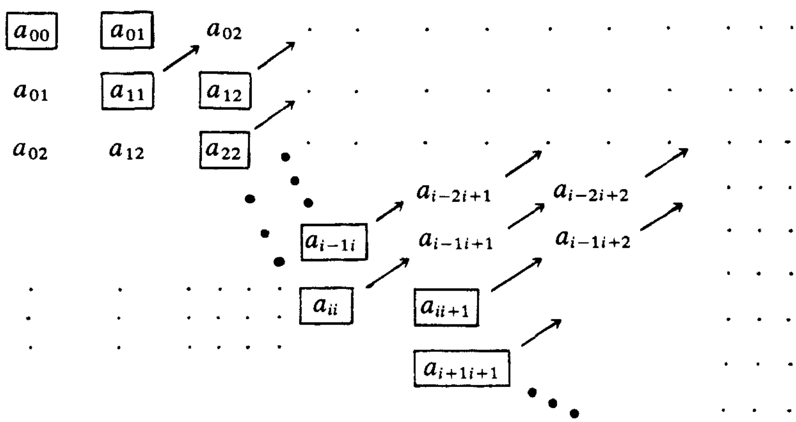
Schematically we have for the symmetric matrix (a_{ik}):

$$(a_{ik}) = \begin{pmatrix} \boxed{a_{00}} & \boxed{a_{01}} & a_{02} & a_{03} \\ a_{01} & \boxed{a_{11}} & \boxed{a_{12}} & a_{13} \\ a_{02} & a_{12} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{03} & a_{13} & a_{23} & \boxed{a_{33}} \end{pmatrix}. \tag{3.30}$$

In equation (3.30) we have indicated by boxes around them those elements that are either determined from inequalities (i.e. the elements in the main diagonal) or are totally arbitrary. Once these have been fixed the remaining follow from equations (3.23)–(3.25).

Having chosen the a_{ik} as described we must verify that they are consistent with the so far unused equation (3.23a)–(3.25a). Take equation (3.23a). Using (3.17) it becomes evident that (3.23a) is transformed into equation (3.23) and so any choice of a_{11} and a_{02} satisfying equation (3.23) also satisfies equation (3.23a). Similar arguments guarantee the consistency of the remaining equations.

It is clear that the general case is no different in principle from this illustration. We may represent the relationships among the a_{ik} thus



where again each of the boxed quantities is arbitrary, except those on the diagonal which are restricted by the conditions

$$D_i^{\pm} > 0 \qquad i = 1, \ldots, n. \tag{3.31}$$

Each of the other a_{ik} is connected by a chain of equalities (indicated by the arrows) to one of the boxed ones and is therefore fixed uniquely. It is important to point out that the parameters s_{ik} are calculated from equations and inequalities which are of first degree in the parameter under consideration. This feature ensures that the determination of a_{ik} can *always* be carried out.

We have, by following the procedure described above, arrived at a decomposition of the moments s_k, as equations (3.17) and (3.19) show

$$s_k = s_k^+ - s_k^- \qquad k = 0, 1, \ldots, 2n-2 \tag{3.32}$$

where $s_k^\pm$ are the moments for two positive moment problems, which is what we set out to do in equation (3.4). The moments $s_k^\pm$ depend on $n-1$ completely arbitrary parameters a_{ii+1}, $i = 0, 1, \ldots, n-2$, and n constrained by virtue of equation (3.31) but otherwise arbitrary quantities a_{ii}, $i = 0, 1, \ldots, n-1$. The total number of arbitrary parameters is thus $2n-1$ which is the same as the number of moments considered in the decomposition (3.32). Since, however, we require piece-wise constant functions σ^+ and σ^- each with n points of increase (cf equation (3.18)) we need in addition the moments s_{2n-1}^+ and s_{2n-1}^-. This means that as well as $s_0, \ldots, s_{2n-2}$ we need to include in our method s_{2n-1} which has not yet been used. As for the solution of the PWMP pertaining to $s_k^\pm$, $k = 0, \ldots, 2n-1$, conditions (3.18), or equivalently (3.21), are necessary and sufficient and since $s_k^\pm$, $k = 0, \ldots, 2n-2$, are calculated by the approach set out in this section, we may define the remaining moments $s_{2n-1}^\pm$ by

$$s_{2n-1} = s_{2n-1}^+ - s_{2n-1}^- \tag{3.33}$$

the choice of (say) s_{2n-1}^+ amounting to a choice of one additional totally arbitrary parameter, making $2n$ in all. Now we can solve the PWMP for s_k^+ and s_k^- and thus obtain σ^+ and σ^-;

$$\int_{-\infty}^{\infty} x^k \, d\sigma^+(x) = s_k^+ \qquad \int_{-\infty}^{\infty} x^k \, d\sigma^-(x) = s_k^- \qquad k = 0, \ldots, 2n-1. \tag{3.34}$$

On defining $\sigma(x) = \sigma^+(x) - \sigma^-(x)$ and using equations (3.32)–(3.34) we deduce that $\sigma(x)$ fulfils equation (3.1) and, hence, solves the NWMP. Clearly $\sigma(x)$ has in general $2n$ points of increase. The case $n \to \infty$ in equation (3.1) is dealt with simply by passing to the limit $n \to \infty$ in equation (3.34) (Akhiezer 1965). We can summarise the procedure presented in this section in the form of the following theorem.

Theorem 3.1. Let $\{s_k\}_0^{2n-1}$ be any sequence of finite and real numbers. The construction of a solution $\sigma(x)$ to equation (3.1) breaks down into four steps.

(i) Calculation of the eigenvalues λ_i of the symmetric matrix $\mathscr{A}$ in (3.8) and determination of the numbers $s_{ik}^\pm$ by (3.16).

(ii) Definition of the quantities $s_k^\pm$ by (3.19) and decomposition of each integer j, $2 \leq j \leq 2n-4$, into two numbers i and k:

$$j = i + k \qquad i = 0, \ldots, n-1 \qquad k = 0, \ldots, n-1. \tag{3.35}$$

If $i_1^{(j)}, k_1^{(j)}, \ldots, i_m^{(j)}, k_m^{(j)}$ are all possible pairs satisfying equation (3.35), set

$$s^+_{i_1^{(j)},k_1^{(j)}} + a_{i_1^{(j)},k_1^{(j)}} = \cdots = s^+_{i_m^{(j)},k_m^{(j)}} + a_{i_m^{(j)},k_m^{(j)}}. \tag{3.36}$$

The superscript (j) implies that equation (3.36) has to be written down for every j.

(iii) Calculation of a_{ii}, $i = 0, \ldots, n-1$, from equations (3.26) and (3.31), whereby by fixing arbitrarily a_{ii+1}, $i = 0, \ldots, n-2$, the remaining parameters are found from equation (3.36).

(iv) Solution of the PWMP for $s_k^\pm$ determined in (i)–(iii). The function $\sigma(x) = \sigma^+(x) - \sigma^-(x)$ is a solution of equation (3.1).

4. Discussion of the solutions

Let us first recall the situation in the standard PWMP. We are given a set of moments $\{s_k'\}_0^{2n-1}$ for which (Akhiezer 1965)

$$\sum_{i,k=0}^{n-1} s'_{i+k} x_i x_k > 0. \tag{4.1}$$

Condition (4.1) is necessary and sufficient for the *unique* determination of the solution $\sigma'(x)$ to the integral equation

$$s'_k = \int_{-\infty}^{\infty} x^k \, \mathrm{d}\sigma'(x) \tag{4.2}$$

$\mathrm{d}\sigma'(x)/\mathrm{d}x$ being of the form

$$\frac{\mathrm{d}\sigma'(x)}{\mathrm{d}x} = \sum_{i=1}^{n} m'_i \delta(x - x'_i) \qquad m'_i > 0. \tag{4.3}$$

Equation (4.3) implies that $\sigma'(x)$ is a piece-wise constant function of x possessing discontinuities m'_i at the points x'_i; for example x'_i and m'_i may correspond to the positions and masses of n particles on the real axis. As long as n is finite the solution $\sigma'(x)$ is unique (Akhiezer 1965). Such a PWMP is called the truncated PWMP. Conversely any n positions x'_i, $i = 1, \ldots, n$, which are completely arbitrary, and n masses m'_i, which must be positive, completely specify $2n$ moments from which the positions and masses may be recovered.

Take now a certain $\sigma(x)$ solving equation (3.1). We can write

$$\sigma = \sigma^{(1)+} - \sigma^{(1)-}. \tag{4.4}$$

The functions $\sigma^{(1)\pm}$ give rise to $s_k^{(1)\pm}$. Assuming that $\sigma^{(1)\pm}$ have n points of increase we have

$$\sum_{i,k=0}^{n-1} s_{i+k}^{(1)\pm} x_i x_k > 0. \tag{4.5}$$

On setting

$$s_{i+k}^{(1)\pm} = s_{ik}^{\pm} + a_{ik}^{(1)} \tag{4.6}$$

and comparing equations (3.18)–(3.19) with equations (4.5)–(4.6) we observe that the procedure of § 3 yields that particular $a_{ik}^{(1)}$ in equation (4.6) and, consequently, $\sigma(x)$ in equation (4.4). Further, according to the theorem concerning the decomposition of $\sigma(x)$ into two non-decreasing $\sigma^\pm(x)$, we may consider in place of equation (4.4), and referring of course to the same σ,

$$\sigma = \sigma^{(2)+} - \sigma^{(2)-}. \tag{4.7}$$

Then by replacing the superscript (1) in equations (4.5)–(4.6) by (2) it is clear that the method of § 3 can also furnish $a_{ik}^{(2)}$. To put it otherwise, by following the approach of § 3 we are capable of constructing a set (though not *the* whole set) of solutions to

(3.1) which are sectionally constant and have in general $2n$ points of increase. Such solutions *exist* and can be found for *any* given set $\{s_k\}_0^{2n-1}$ with finite n. This NWMP can be called (in analogy to the relevant PWMP) the truncated NWMP. It is worth noticing that the degree of arbitrariness in decomposing the truncated NWMP into two truncated PWMP is exactly the same as that of defining one positive weight distribution, since the parameters a_{ii+1}, s_{2n-1}^{+} and a_{ii} can be taken to correspond to x_i' and m_i' respectively.

The procedure of § 3 includes as a special case the PWMP. In fact, if all the eigenvalues λ_i of $\mathscr{A}$ in equation (3.10) are positive, Q in equation (3.5) is positive definite thus showing that $\{s_k\}_0^{2n-1}$ constitute the moments of a PWMP. Moreover, if $\mathscr{A}$ has a zero eigenvalue then we have yet again a PWMP, its solution possessing *less* than n points of increase. This is so because the following theorem holds.

Theorem 4.1. Let a sequence $\{s_k'\}_0^{2n-1}$ be given. The necessary and sufficient conditions for the existence of a non-decreasing piece-wise constant function $\sigma'(x)$ fulfilling equation (4.2) and having n', $n'<n$, points of increase are

$$D_k' = \begin{vmatrix} s_0' & s_1' & \dots & s_k' \\ s_1' & s_2' & \dots & s_{k+1}' \\ \vdots & \vdots & & \vdots \\ s_k' & s_{k+1}' & \dots & s_{2k}' \end{vmatrix} \begin{matrix} >0 & \quad \text{for } k=0,\dots,n'-1 \\ =0 & \quad \text{for } k=n',\dots,n,\dots . \end{matrix} \tag{4.8}$$

Proof. Assume the existence of a $\sigma'(x)$ as specified in the theorem. By a straightforward application of the standard theory of determinants we obtain, on denoting with x_i' and m_i', $i=1,\dots,n$, the points of increase and discontinuities respectively and utilising equations (4.2)–(4.3)

$$D_0' = \sum_{i=1}^{n'} m_i' \qquad D_1' = \sum_{i_1<i_2}^{n'} m_{i_1}' m_{i_2}' (x_{i_1}' - x_{i_2}')^2$$

$$\begin{aligned} D_k' = \sum_{i_1<i_2<\dots<i_{k+1}}^{n'} & m_{i_1}' \dots m_{i_{k+1}}' (x_{i_1}' - x_{i_2}')^2 \dots (x_{i_1}' - x_{i_{k+1}}')^2 \\ & \times (x_{i_2}' - x_{i_3}')^2 \dots (x_{i_2}' - x_{i_{k+1}}')^2 \dots (x_{i_k}' - x_{i_{k+1}}')^2 \\ D_{n'-1}' = m_1' m_2' & \dots m_{n'}' (x_1' - x_2')^2 \dots (x_1' - x_{n'}')^2 \dots (x_2' - x_3')^2 \dots \\ & \times (x_2' - x_{n'}')^2 \dots (x_{n'-1}' - x_{n'}')^2 \end{aligned} \tag{4.9}$$

$$D_k' = 0,\ k = n', \dots, n, \dots . \tag{4.10}$$

If $n'=n$, equation (4.10) becomes $D_k'=0$, $k=n, n+1, \dots .$

From equations (4.9)–(4.10) and since $m_i'>0$ the necessity of conditions becomes obvious. The proof of sufficiency proceeds along the same lines as the verification of the sufficiency of conditions (3.21) (Akhiezer 1965). It should be noted that conditions (4.8) are equivalent to

$$\sum_{i,k=0}^{n-1} s_{i+k}' x_i x_k \geq 0. \tag{4.11}$$

An interesting case yet to be considered is the following. Choose, on the real axis, n positions x_i and fix n arbitrary (*not* necessarily positive) quantities m_i. Then we can calculate a set of moments $\{s_k\}_0^{2n-1}$ as

$$s_k = \sum_{i=1}^{n} m_i x_i^k \qquad k = 0, 1, \ldots, 2n-1. \tag{4.12}$$

The determinants

$$D_k = \begin{vmatrix} s_0 & s_1 & \ldots & s_k \\ s_1 & s_2 & \ldots & s_{k+1} \\ \vdots & \vdots & & \\ s_k & s_{k+1} & \ldots & s_{2k} \end{vmatrix} \qquad k = 0, 1, \ldots, n-1 \tag{4.13}$$

can again be written in the form of equation (4.9). Thus it becomes evident that $D_k \neq 0$. The calculation of moments s_j, $j \geqslant 2n$, does not furnish us with any additional information since, as in equation (4.10), $D_k = 0$ for $k \geqslant n$. At this point it is worth noting that the omission of the words 'sufficient' and 'non-decreasing' renders theorem 4.1 valid also for arbitrary m'_i, whereby for $k = 0, \ldots, n'-1$ we have only $D'_k \neq 0$ as now $\sigma'(x)$ is not necessarily non-decreasing. Clearly the function $\sigma_1(x)$ defined by

$$\frac{d\sigma_1(x)}{dx} = \sum_{i=1}^{n} m_i \delta(x - x_i) \tag{4.14}$$

is a solution to

$$s_k = \int_{-\infty}^{\infty} x^k \, d\sigma(x). \tag{4.15}$$

Conversely, if we are given a set $\{s_k\}_0^{2n-1}$ and we somehow know or suspect that they have been constructed as those in equation (4.12), we can apply the standard procedure used in the PWMP, where $m_i > 0$, and recover the x_i and the m_i. That procedure works simply because the structure of the moments in equation (4.12) is mathematically *identical* to that of the moments constituting a PWMP, as a glance at equations (4.2)–(4.3) reveals. But since the determinants D_k in equation (4.13) can be cast into the form (4.9), and because the m_i are arbitrary, then $D_k \gtrless 0$, which implies that the quadratic form $Q = \Sigma\, s_{i+k} x_i x_k$ is not positive definite. Thus the set $\{s_k\}_0^{2n-1}$ belongs to a NWMP. Therefore the procedure of § 3 is applicable and, hence, we shall get a set of solutions to equation (4.15), i.e. to equation (3.1), which possess in general $2n$ points of discontinuity, in contrast to $\sigma_1(x)$ in equation (4.14) that has precisely n discontinuities. This situation is another manifestation of the non-uniqueness of the NWMP and, consequently, of its most important difference with the PWMP. In contrast to the PWMP in the NWMP n positions x_i and masses $m_i \gtrless 0$ do *not* uniquely specify a distribution function $\sigma(x)$.

In practice, when we are given a set $\{s_k\}_0^{2n-1} = S$, we must check first by using the method of the PWMP if S can be written as in equation (4.12). If in the course of such an investigation a non-real x_i turns up, it is immediately evident that S cannot be attributed to a $\sigma_1(x)$ defined by equation (4.14). Then the approach of § 3 must be applied.

There still remain two possibilities to be mentioned. First, if the matrix $\mathscr{A}$ in § 3 has no positive eigenvalues then $Q = \Sigma\, s_{i+k} x_i x_k \leqslant 0$, which shows that the set of s_k pertain to a $\sigma(x)$ which is *non-increasing*; such a case is exactly the opposite of the PWMP and an interesting feature here is that the determinants (4.13) alternate in sign, as equation (4.9) shows: $D_0<0, D_1>0, D_2<0, \ldots, D_{n'-1}=|D_{n'-1}|(-1)^{n'}, D_{n'}=\cdots=D_n=0=\cdots$ (note that $m_i<0$ and that n', $n'\leq n$, are the points of *decrease* of $\sigma(x)$).

Second, if there exists a piece-wise constant function $\sigma_N(x)$ having $N>n$ discontinuities and possessing the above set of moments S, then that $\sigma_N(x)$ cannot be reproduced by the standard method of the PWMP because, in that case, we also need the moments $s_{2n}, s_{2n+1}, \ldots, s_{2N-1}$; this situation corresponds to the usual PWMP where $\{s'_k\}_0^{2n-1}$ suffice only for the unique determination of a non-decreasing $\sigma'(x)$ with (at most) n discontinuities. A remarkable feature of the procedure of § 3, however, is that it enables us to find sectionally constant functions $\sigma(x)$ having in general $2n$ discontinuities and, thus, by appropriately choosing the a_{ik} we may (though not always) be able to calculate a $\sigma(x)$ having all or some of the remaining moments $s_{2n}, \ldots, s_{2N-1}$.

5. Summary

In this work motivated by investigations in many-body quantum physics we have presented a well defined procedure for the construction of solutions to equation (3.1). The solutions obtained constitute an infinite set and possess a high degree of arbitrariness which is contained in precisely $2n$ canonical parameters. The discussion concerning the above solutions has revealed the differences between the standard PWMP and the NWMP. Finally the procedure introduced in § 3 may serve as the means for gaining an insight into the physics inherent in equation (2.8) which is related to eigenvalue calculations in various problems.

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